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Fuzzy Space-Time Geometry and Particle's Dynamics

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Abstract The quantum space-time with Dodson-Zeeman topological structure is studied. In its framework, the states of massive particle m correspond to the elements of fuzzy ordered set (Foset), i.e. the fuzzy points. Due to their partial ordering, m space coordinate x acquires principal uncertainty σ_x. Schroedinger formalism of Quantum Mechanics is derived from consideration of m evolution in fuzzy phase space with minimal number of additional axioms. The possible particle's interactions on fuzzy manifold are studied and shown to be gauge invariant.

Keywords Fuzzy geometry · Quantization · Fuzzy sets

1 Introduction

The hypothetic structure of space-time at microscopic (Plank) scale and its relation to axiomatic of Quantum Mechanics (QM) is actively discussed now [1–3]. In particular, it was proposed that such fundamental properties of space-time manifold M_{ST} as its metrics and topology can differ significantly at Planck scale from standard Riemanian formalism [4]. In this vein, it's worth to explore also the indications from Sets Theory and consider the various possible structures of M_{ST} fundamental set [5, 6]. For example, Posets and the fuzzy ordered sets (Fosets) were used for the construction of different variants of the novel Fuzzy Geometry [7, 8]. Hence it's instructive to study what kind of physical theory such geometry induces [1, 5]. It was found that in its framework the quantization by itself can be defined as the transition from the ordered phase space to fuzzy one, i.e. the quantum properties of particles and fields are induced directly by Fuzzy Geometry of their phase space and don't need to be postulated separately of it [1]. In our previous papers as the simple example of such transition the quantization of nonrelativistic particle was regarded; it was shown that Fuzzy Geometry induces the particle's dynamics which is equivalent to Schroedinger QM dynamics [1, 2]. For the comparison is worth to mention here the extensive studies of

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noncommutative fuzzy spaces, both finite (sphere, tori) and infinite ones [9, 10]. Besides the standard QM noncommutativity of x, p_x observables, such theories postulate that the orthogonal space coordinates x, y, z also don't commute. In such terms we study the commutative fuzzy geometry, because in our formalism the noncommutativity of any pair of phase space coordinates isn't assumed beforehand,it can appear (or not) only as the result of derivations. Here we shall investigate mainly the novel symmetries of fuzzy manifold and related physical states. The resulting constraints on the interactions between particles are obtained; it will be shown that such interactions are gauge invariant and under simple assumptions correspond to Yang-Mills fields.

Remind that in 1-dimensional Euclidean Geometry, the elements of its manifold X are the points x_a which constitute the ordered set. For the elements of partially ordered set (Poset) $D = \{d_i\}$, beside the standard ordering relation between its elements $d_k \leq d_l$ (or vice versa), the incomparability relation $d_k \wr d_l$ is also permitted; if it's true, then both $d_k \leq d_l$ and $d_l \leq d_k$ propositions are false. To illustrate its meaning, consider Poset $D^T = A \cup B$, which includes the subset of 'incomparable' elements $B = \{b_j\}$, and the ordered subset $A = \{a_i\}$. Let's settle that in A the element's indexes grow correspondingly to their ordering, so that $\forall i, a_i \leq a_{i+1}$. Any b_j is incomparable at least to one a_i. As the example, let's consider some interval $\{a_l, a_{l+n}\}$ and suppose that $b_j \in \{a_l, a_{l+n}\}$, i.e. $a_l \leq b_j$; $b_j \leq a_{l+n}$ and $b_j \wr a_i$; iff $l + 1 \leq i \leq l + n - 1$. In this case, b_j in some sense is 'smeared' over $\{a_l, a_{l+n}\}$ interval. To introduce the fuzzy relations, let's put in correspondence to each b_j, a_i pair the weight $w_i^j \geq 0$ with the norm $\sum_i w_i^j = 1$. Under this conditions D^T is Foset, b_j called the fuzzy points. The simple example is the homogeneous incomparability: $w_i^j = \frac{1}{n}$ for $a_i \in [a_l, a_{l+n}]$; $w_i^j = 0$ outside of it. The continuous 1-dimensional Foset C^F is defined analogously; $C^F = B \cup X$ where B is the same as above, X is the continuous ordered subset, which is equivalent to R^1 axis of real numbers. Fuzzy relations between b_j, x_a are described by the positive distribution $w^j(x_a) \geq 0$ with the norm $\int w^j dx = 1$. Note that in Fuzzy Geometry $w^j(x)$ doesn't have any probabilistic meaning but only the algebraic one.

The particle's state in 1-dimensional Classical Mechanics is described as the ordered point $x(t)$ in X. Analogously to it, in 1-dimensional Fuzzy Mechanics (FM) the particle m corresponds to the fuzzy point $b(t)$ in C^F; it characterized by the positive density $w(x, t)$ with norm $\|w\| = 1$. However, FM doesn't exclude the existence of other m degrees of freedom on which m evolution can depend. To account them it is supposed that in arbitrary reference frame (RF) m is described by a fuzzy state $|g(t)\}$, and all g components are the real functions of one or more coordinates, i.e. are the local, bilocal, n-local fields:

$$\{g_i^1(x)\}; \quad i = 1, l_1; \qquad \{g_j^2(x, x')\}; \quad j = 1, l_2, \ldots, \text{etc.};$$

where $g_1^1(x) = w(x)$. In distinction from QM, $|g\}$ states set M_s isn't postulated to be the linear space of any kind *a priori*. FM supposedly is invariant relative to the space and time shifts and space/time reflections. In our approach, FM is constructed as the minimal theory, i.e. the number of $|g\}$ degrees of freedom and free parameters is as minimal as necessary for the theory consistency.

2 Fundamentals of Fuzzy Evolution

We shall not describe here the complete derivation of FM formalism which can be found in [2], featuring here only its most important steps. First, the effect of source smearing (SS) should be regarded, which is the essence of FM distinction from classical statistical mechanics. It's natural to assume that in FM, alike in any consistent dynamical model, a localized state g_0 with $w_0(x) \sim \delta(x - x_0)$ evolves freely to the wide distribution $w(x,t)$ with dispersion $\sigma(t) \to \infty$ at $t \to \infty$. For illustration, one can regard the simple toy-model of FM in which the evolution of point-like initial m state (source) g_0 in 1-dimension is described by the classical diffusion propagator [12]; for m located in $x = 0$ at $t = 0$:

$$w(x,t) = \Gamma_D(x,t) = \frac{1}{2\kappa\sqrt{\pi t}} \exp^{-\frac{x^2}{4\kappa^2 t}} \tag{1}$$

where κ is the diffusion constant. Yet for any nonlocalized state g_0 its evolution differs from the classical diffusion. To demonstrate it, let's consider 1-dimensional analogue of QM two slits experiment (TSE) [13]. This is the system of $n = 2$ initial m states $g^0_{1,2}$ located in small bins $Dx_{1,2}$ which centers are in $x_{1,2}$.

Consider first TSE for the probabilistic mixture g_0^{mix} of $g^0_{1,2}$ initial m states localized in $x_{1,2}$ with probability w_i^0. In each event m is emitted definitely by the bin Dx_1 or Dx_2, therefore g_0^{mix} structure is described by the proposition:

$$LP^{mix} := m \in Dx_1 \vee m \in Dx_2$$

The resulting m distribution over this ensemble at any $t > t_0$ will be:

$$w_{mix}(x,t) = w_1(x,t) + w_2(x,t) = \sum w_i^0 \Gamma_D(x - x_i, t)$$

In our FM consideration there are important only the configurations for which $w_{1,2}(x,t)$ intersect largely, i.e. for $L_x = |x_1 - x_2|$ it should be $L_x \leq \sigma_x(t)$ where $\sigma_x(t)$ is $w_{1,2}$ dispersion (for our toy-model $\sigma(t) \sim \kappa t^{\frac{1}{2}}$). Now suppose that TSE performed on the fuzzy (pure) state $|g_0^s\}$ for which m coexists simultaneously in both bins Dx_i with the weights w_i^0. Its structure is described by the proposition:

$$LP^s := m \wr Dx_1 \wedge m \wr Dx_2$$

Plainly, LP^{mix} and LP^s are incompatible: $LP^{mix} \wedge LP^s = \emptyset$. So if to rewrite w_s as:

$$w_s(x,t) = w_p(x,t) + k_w w_{mix}(x,t) \tag{2}$$

where $w_p \geq 0$ is arbitrary, then it follows that $k_w = 0$, i.e. any w_{mix} content in w_s is excluded. If w_s, w_{mix} supports in X mainly coincide, such k_w value is possible only if in one or more points x_j where $w_{mix}(x_j) \neq 0$ we have $w_s(x_j) = 0$. This is the necessary and sufficient condition of w_s nonclassicality. Since w_s, w_{mix} norms are the same, it means that w_s oscillates around w_{mix}, so such distribution describes the interference pattern similar to the ones observed for QM superpositions.

By itself, Fuzzy Geometry doesn't contain any length parameters on which $\sigma_x(t)$ can depend. Therefore in nonrelativistic case, one can omit them at all and choose minimal FM ansatz with $\sigma_x(t) \sim \infty$ for point-like source g^0 at any $t > 0$, it corresponds to m free evolution with the maximal fuzziness. Hence at $x \to \pm\infty$, $\lim w(x - x_i, t) \neq 0$ (or the limits don't exist); this w property is called x-limit condition (x-LC). Thus, $w(x,t)$ is Schwartz distribution (generalized function) with infinite support and undefined $\bar{x}$ [14]. Such $|g\}$ evolution seems quite exotic, however, in QM the point-like state evolves analogously [13]. For $|g_0^s\}$ superposition (2) of two point-like states $g^0_{1,2}$ the resulting $w_s(x,t)$ should also satisfy to x-LC. Such $w_s(x)$ responds to the maximal SS, i.e. the maximal loss of information about

m origin [5]. Meanwhile, $w'_s(x,t) = w_s(x + a_x, t)$ also corresponds to it for any arbitrary a_x, because SS depends on w_s form only whereas $\bar{x}$ is undefined. Hence w'_s also can be the result of free evolution of some $|g'_0\}$ state localized in two bins in $x_{1,2}$. It evidences that beside $w(x)$, the fuzzy states $|g\}$ include at least one more degree of freedom. Since a_x depends on $|g'_0\}$ both in x_1 and x_2, it should be represented by the bilocal $|g\}$ component $g_1^2(x_1, x_2)$ defined above. In minimal FM g_1^2 is the real function of two variables $g_1^2 = K(x, x')$, for which we choose the gauge: $K(x,x) = 0$. If to fix some arbitrary x_c, then K can be expressed as the function of one observable $\alpha(x) = K(x, x_c)$. So for free m evolution g can be treated as the local field $E^g(x) = \{w(x), \alpha(x)\}$. It can be mapped to the symmetric $|g\}$ ansatz represented by the complex function $g(x) = w^{\frac{1}{2}}(x)\exp ic_\alpha\alpha(x)$, where c_α is the fixed parameter. In this case, w/g zero-equivalence holds: $w(x) = 0 \Leftrightarrow g(x) = 0$ and the same is true for w, g limits at $x \to \infty$. Thus, if x-LC holds for given $w(x,t)$, then it should be so also for corresponding $g(x,t)$. Note that the analogous to $K(x,x')$ bilocal structure possesses QM density matrix $\rho(x,x')$. In principle, the effects of $|g\}$ bilocality can become important for the interactions between particles and they will be regarded below in detail.

In the regarded framework, the evolution of complex $g(x,t)$ is described by the unitary operator $\hat{U}(t)$, so that: $|g(t)\} = \hat{U}(t)|g_0\}$. It possesses the properties of group element: $U(t_1 + t_2) = U(t_1)U(t_2)$, therefore it can be expressed as $\hat{U}(t) = e^{-i\hat{H}_0 t}$ where $\hat{H}_0$ is an arbitrary constant operator [11]. It isn't supposed beforehand to be linear, but our analysis has shown that only linear H_0 correspond to physically interesting solutions [2]. The free g evolution is invariant relative to X shifts performed by the operator $\hat{W}(a) = \exp(a\frac{\partial}{\partial x})$. Because of it, $\hat{U}(t)$ should commute with $\hat{W}(a)$ for the arbitrary a. It's equivalent to the relation $[\hat{H}_0, \frac{\partial}{\partial x}] = 0$, from which follows that for g Fourier transform $\varphi(p,t)$ the operator $\hat{H}_0$ is an arbitrary function of p: $H_0 = F_0(p)$. Consider now the free evolution of point-like state $g_0(x, t_0) = \delta(x - x_0)$. Its Fourier transform $\varphi_\delta(p) = \exp(ipx_0)$ and it follows that:

$$\varphi(p,t) = U(t)\varphi_\delta = e^{-iF_0(p)(t-t_0)+ipx_0}$$

below $x_0 = 0, t_0 = 0$ settled. The transition $\delta(x) \to g_u(x,t)$ develops continuously without breaking points if $g_u(x,t_j)$ constitutes δ-sequence, i.e. $g_u(x,t_j) \to \delta(x)$ for any sequence $\{t_j\} \to +0$ [12]. This condition is fulfilled only if $g_u(x,t)$ has $t = 0$ pole, and the substitution $z = \frac{x}{f(t)}$ results in:

$$g_u(z,t) = \frac{1}{f(t)}e^{i\gamma(z)},$$

where $f(t) \to 0$ at $t \to +0$, $\gamma(z)$ is arbitrary complex function. If in this limit the asymptotic relation:

$$\int_{-\infty}^{\infty} g_u(z,t)f(t)dz \to 1$$

is also fulfilled, then under these conditions $g_u(x,t) \to \delta(x)$ at $t \to +0$. After z substitution g_u Fourier transform can be expressed also as:

$$\varphi(p,t) = c_0\int_{-\infty}^{\infty} dz e^{i\gamma(z)+izpf(t)} = \exp^{-i\Gamma[pf(t)]} \tag{3}$$

From that it follows: $F_0(p) = \frac{p^s}{2m_0}$, $f(t) = d_r t^r$, where $rs = 1$, m_0, d_r are arbitrary real parameters [2]. If $H_0 = F_0(p)$ is regarded as m free Hamiltonian, then from its symmetry properties and the energy positivity it follows that $m_0 > 0$ and the consistent s values are only the natural even numbers. For $s = 2$ it gives $g_u(x,t) \neq 0$ at $x \to \pm\infty$, i.e. g_u satisfies to x-LC, as minimal FM ansatz demands. Yet it is violated for H_0 with $s \geq 4$, hence such solutions don't suit to minimal FM ansatz and should be rejected [2].

Note that the point-like state $g_0(x)$ regarded here has the undefined norm $\|w\|$ and so it's more correct to exploit as the localized g state w-normalized function $\eta_\delta(x)$ called also the squire root of $\delta(x)$ [15]. However, H_0 ansatz obtained above can't change after this renormalization, because it is, in fact, just the multiplication of $g_0(x)$ and $g_u(x,t)$ on the infinitesimal constant. Meanwhile, $g_u(x,t)$ describes FM free propagator for particle with mass m_0, so an arbitrary normalized function $g_{in}(x)$ will evolve as:

$$g(x',t) = \int g_u(x'-x,t) g_{in}(x) dx = \sqrt{\frac{m_0}{-i2\pi t}} \int e^{\frac{im_0(x'-x)^2}{2t}} g_{in}(x) dx \tag{4}$$

which coincides with the free g_{in} evolution in QM formalism [13].

Overall, the free evolution of m fuzzy state $|g(t)\}$ is described by Schroedinger equation with free Hamiltonian $\hat{H}_0 = \frac{\hat{p}^2}{2m_0}$. In this framework, the normalized function $g(x)$ admits the orthogonal decomposition on $|x_a\rangle = \delta(x - x_a)$, so $|x_a\rangle$ set constitutes the complete system [11]. $|g\}$ set M_s is equivalent to the complex rigged Hilbert space $\mathcal{H}$ with the scalar product $g_1 * g_2 = \int g_1^* g_2 dx$. In FM x is m observable and it's sensible to suppose that $\hat{p}$ and any self-adjoint operator functions $\hat{F}_Q(x,p)$ are also m observables. So in FM all physical observables constitute the linear algebra of self-adjoint operators (Segal algebra). Generalization of FM formalism on 3 dimensions is straightforward and doesn't demand any serious modification of described ansatz. The only novelty is that $|g\}$ bilocal correlation $K(\vec{r}_1, \vec{r}_2)$ should be independent of the path l between $\vec{r}_{1,2}$.

Note that Planck constant $\hbar = 1$ in our FM calibration, alike it's done in Relativistic QM; in FM framework, it only relates x, p scales and doesn't have any other meaning [13]. The proposed FM considers the nonrelativistic particle for which x is the fuzzy coordinate, yet one can choose any observable q as the fundamental fuzzy coordinate. It can be especially important in relativistic case where $\vec{r}$ supposedly can't be the proper observable [15]. For the relativistic free evolution the linearity of state evolution becomes the important criterion for the choice of consistent ansatz. For massive particle m the minimal solution is 4-spinor $g_i(\vec{r},t)$; $i = 1,4$, its evolution is described by Dirac equation for spin-$\frac{1}{2}$, hence such particle is fermion. FM approach, in principle, can be extended on quite different physical systems, besides the single particle. In particular, it can be Fock space of the secondary quantization, in this case, the occupation numbers for particle's states $N_c(\vec{p})$ can be regarded as the fuzzy values.

3 Particles and Fields Interactions in FM

Now we shall consider the interaction between fuzzy states in nonrelativvstic FM and attempt to extend the obtained results on relativistic case. Note first that by derivation FM free Hamiltonian H_0 induces $\mathcal{H}$ dynamical asymmetry between $|\vec{r}\rangle, |\vec{p}\rangle$ 'axes' which is absent in standard QM formalism. By itself, FM description of physical systems doesn't need the corresponding classical system as the starting point [15]. Hence in our approach only some general properties of classical dynamics will be implemented but no particular correspondence principle is used. In this case, the bilocality of fuzzy state $|g\}$ will be important, it can be accounted if to represent $|g\}$ by the density matrix:

$$\rho(\vec{r},\vec{r'}) = [w(\vec{r})w(\vec{r'})]^{\frac{1}{2}} e^{iK(\vec{r},\vec{r'})}$$

- where $K = \alpha(\vec{r}) - \alpha(\vec{r'})$; we denote ρ set of pure states as $\mathcal{M}$. Let's consider in this framework the interaction of particles m and M in the limit $M \to \infty$ for M state with $w(\vec{R}, t_0) \sim \delta(\vec{R})$, i.e. M state is practically classical and its change after finishing of m, M interaction is insignificant. It is reasonable to suppose that in this limit the operator of m, M interaction H_{int} factorized from the free m evolution, so that:

$$i\frac{\partial \rho}{\partial t} = \hat{F}_0(\rho) + \hat{H}_{int}(\rho) \tag{5}$$

where $\hat{F}_0$ corresponds to H_0 for ρ. H_{int} acts on bilocal state ρ and so, in general, can be also bilocal operator. In the simplest case, H_{int} is bilocal scalar field: $H_{int} = QB(\vec{r},\vec{r'})$ where Q is m dynamical charge. From ρ, H_{int} symmetry relative to $\vec{r} \leftrightarrow \vec{r'}$ it follows that

$$B = V(\vec{r}, t) - V(\vec{r'}, t) \tag{6}$$

where V is defined up to $f(t)$ gauge. However, in FM V isn't related *a priory* to any classical potential but is only the formal decomposition of bilocal field B. Naturally, H_{int} describes by this formulae m interaction with the classical potential V_{cl} also [11]. From (5) it follows that $K(\vec{r},\vec{r'})$, $w(\vec{r})$ infinitesimal evolution for any initial $\rho_{in}(t_0)$ can be factorized as:

$$dK = dK_0 + H_{int}dt \tag{7}$$

$$dw = dw_0 + O(dt^2) \tag{8}$$

where dK_0, dw_0 are K, w variations for free m evolution and given ρ_{in}. In fact, the regarded $dw(\vec{r})$ is stipulated by $K(t_0)$ via m flow conservation: $\frac{\partial w}{\partial t} = \operatorname{div}\vec{j}$, because $\vec{j}$ depends on $K(t_0)$ and $w(t_0)$ but not on H_{int} directly. It means that by itself such $w(\vec{r})$ evolution during every small time gap $\{t, t+dt\}$ differs insignificantly from the free one, meanwhile, each initial $K(t)$ depends on H_{int}. The nature of such bilocal interaction is similar to EPR-Bohm nonlocality [11]: m simultaneously is presented in $\vec{r}$ and $\vec{r'}$, so it can be sensitive to B in two distant points without violation of causality. Such bilocal H_{int} supposedly acts on bilocal g component K and doesn't influence the local density $w(\vec{r})$ directly. Such ansatz describes the minimal implementation of interactions in FM formalism which occurs via K perturbation by H_{int}. Yet it conserves the general mechanism of m free FM evolution and, in particular, the dynamical $\mathcal{H}$ asymmetry, or more precisely $\mathcal{M}$ asymmetry. In this framework, such perturbation of m free motion can be regarded also as the local $\mathcal{M}$ deformation by H_{int}.

Now let's discuss the possible relativistic description of such interactions. H_{int} describes m, M interaction energy factorized from m kinetic energy. So the corresponding $V(\vec{r})$ should be 4-th component A_0 of some 4-vector $A_\mu(\vec{r})$ and H_{int} is expressed as the difference of two such components. Hence such interaction is accounted covariantly by substituting: $\partial_\mu \to \partial_\mu + QA_\mu$ in Dirac equation for $g_i(\vec{r}, t)$ [16]. As follows from (5) its bilocal ansatz is described by Dirac equation for 4-spinor density matrix. As easy to see, our simple model reproduces the main features of Quantum Electrodynamics, if Q is regarded as m electric charge. Consequently, FM would possess $U(1)$ local gauge invariance under assumption that free A_μ field also obeys it. However, it doesn't need the hypothesis of local gauge invariance of m quantum phase [15]. The preliminary results for the interactions of fermion multiplets show that in such theory their interactions also possess $SU(n)$ gauge invariance and transferred by the corresponding Yang-Mills fields [16].

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of Set Theory and Topology together with the natural assumptions about systems evolutions. It allows to suppose that the quantization phenomenon has its roots in foundations of mathematics and logics [11]. The main aim of FM, as well as other studies of fuzzy spaces, is the construction of nonlocal QFT (or other more general theory) [10, 15]. In this vein, FM provides the interesting opportunities, being generically nonlocal theory which, in the same time, is Lorentz covariant and manifests the gauge invariance.

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